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Storage capacity of a diluted neural network with Ising couplings

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Abstract. The capacity for storing random patterns in a diluted neural network is determined following the method of Gardner. The non-zero coupling coefficients are restricted to take on the Ising values ± 1. Different degrees of dilution are considered. The maximum value $\alpha_c = 1.17$ for the storage capacity is obtained when each neuron is connected to 63% of the other neurons.

1. Introduction

The study of the storage capacity of a neural network has progressed strongly in recent years, due mainly to the pioneering work of Gardner [1]. Adapting some methods from the theory of spin glasses, she succeeded in determining the storage capacity of a neural network when the coupling coefficients J_{ij} are allowed to take on unrestricted values except for an overall normalisation condition (spherical model). Using N to denote the number of neurons and p the number of memorised patterns and using α for the storage ratio p/N, the storage capacity α_c is defined as the maximum value of α, i.e. the maximum number of patterns which can be stored per neuron. For the spherical model Gardner obtained $\alpha_c = 2$, in agreement with earlier derivations by Cover [2] and Venkatesh [3]. The method was extended and reformulated in the language of statistical mechanics by Gardner and Derrida [4] and also applied to the case where the coupling coefficients are restricted to take on the values ± 1 only (Ising couplings). For this case, they obtained $\alpha_c = 4/\pi$ but immediately remarked that this value is too large for the exact storage capacity. Gardner and Derrida found that this failure is caused by the lack of stability of the replica-symmetric saddle point in the case of Ising couplings.

In a recent paper, Krauth and Mézard [5] have reinvestigated the problem of storage capacity of a network with Ising couplings. Using a first step of replica-symmetry breaking, they obtained $\alpha_c \cong 0.83$ for the storage capacity, in good agreement with numerical calculations. More interesting, they obtained the same value from the replica-symmetric calculation by requiring that the quenched entropy must be non-negative. This condition should clearly be satisfied because the space of interactions contains a countable number of coupling vectors in the case of Ising couplings.

In this paper, we extend the calculation of Krauth and Mézard to the case of a diluted network with Ising couplings. Each neuron is coupled to the same fraction f of the other neurons. Unlike most calculations for diluted networks where the choice of couplings which are cut is random [6], we consider the problem of how to maximise the total number of stored patterns under the constraint that each neuron be coupled to just fN other neurons. In this approach, the choice of disconnected couplings is selective and strongly correlated to the stored patterns. A similar approach has been followed for the Hopfield model by Sompolinsky [7] and by Van Hemmen [8]. For $f=1$, we recover the calculation of Krauth and Mézard. We are interested in studying how the storage capacity $\alpha_c(f)$ depends on the degree of dilution. In section 2, we calculate the quenched entropy using the replica-symmetric ansatz. In section 3, we obtain an upper bound for $\alpha_c(f)$ from the annealed entropy. In section 4, we study the solution of the stationary point equation and determine $\alpha_c(f)$ from the non-negativity of the quenched entropy. The results are discussed in the last section.

2. Calculation of the quenched entropy

We will assume that the reader is familiar with the papers of Gardner [1] and of Krauth and Mézard [5]. In this section, we will closely follow the reasoning of Gardner but will use the simpler notation of Krauth and Mézard.

We consider a network of $N+1$ neurons which are labelled by the index $i=0,1,\ldots N$. Each neuron i is coupled to the other N neurons with coupling coefficients J_{ij} which can take on the values ± 1 or 0. We focus our attention on the neuron $i=0$ and use the notation $J_j = J_{0_j}$. The N coupling coefficients J_j can be considered as the components of an N-dimensional coupling vector $\boldsymbol{J}$ in Gardner's phase space of interactions. We fix the number of zero couplings and call it Z. This yields the following normalisation condition for the coupling coefficients:

$$\sum_{j=1}^{N} J_j^2 = \boldsymbol{J}\cdot\boldsymbol{J} = N - Z = fN \tag{1}$$

where f is the fraction of non-zero couplings. The coupling coefficients J_j must now be chosen in such a way that the p random patterns $\{\xi_j^\mu\}$ are memorised. This yields the usual p conditions [5]

$$\frac{1}{\sqrt{Nf}}\sum_{j=1}^{N} J_j\eta_j^\mu = \frac{1}{\sqrt{Nf}}\boldsymbol{J}\cdot\boldsymbol{\eta}^\mu > K \qquad \mu = 1,\ldots,p \tag{2}$$

where $\eta_j^\mu = \xi_0^\mu\xi_j^\mu$ and K is a stability parameter ($K \geqslant 0$). Our problem is to determine, for any given value of $f \in [0,1]$, the maximum number of random patterns for which a solution for J_j exists which satisfies all conditions (1) and (2).

Following the original ideas of Gardner [1], we calculate the quantity

$$S_{\mathrm{Q}} = \lim_{N\to\infty}\frac{1}{N}\langle\log\Omega\rangle \tag{3}$$

which will be called the quenched entropy S_{Q}. The brackets $\langle\ \rangle$ represent the average over the p random patterns $\{\xi_j^\mu\}$ and Ω is the number of coupling vectors $\boldsymbol{J}$ which satisfy the $p+1$ conditions (1) and (2). This number is given by the expression

$$\Omega = \sum_{\{J_j\}}\prod_{\mu=1}^{p}\theta\left[\frac{1}{\sqrt{Nf}}\boldsymbol{J}\cdot\boldsymbol{\eta}^\mu - K\right]\delta_{\mathrm{Kr}}(\boldsymbol{J}^2, fN). \tag{4}$$

Each coupling coefficient J_j in the sum takes on the values ± 1 and 0 independently, the Kronecker delta taking care of condition (1).

From its definition, the value of Ω is a non-negative integer. For given N, when the number p of patterns increases, the number of conditions (2) increases so that the value of Ω must decrease. As long as all conditions (1), (2) can be satisfied, Ω will be larger than or equal to 1 and S_Q is positive. The critical storage capacity α_c is reached when S_Q becomes equal to zero.

The calculation of S_Q is done using the replica method [1]. The θ functions are represented by their Fourier integral and the Kronecker delta is similarly expressed as a sum of exponentials by using

$$\delta_{Kr}(k, m) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp[i(k-m)\psi]\, d\psi. \tag{5}$$

Using the straightforward techniques of Gardner [1], we obtain

$$\langle\Omega^n\rangle = \prod_{1\le a\le n}\left(\int_{-\pi}^{\pi}\frac{d\psi_a}{2\pi}\right)\prod_{1\le a<b\le n}\left(\int_{-\infty}^{\infty}\frac{dF_{ab}}{(2\pi/Nf)}\int_{-1}^{1} dq_{ab}\right)\exp[NG_n(\psi_a, F_{ab}, q_{ab})] \tag{6}$$

where

$$G_n(\psi_a, F_{ab}, q_{ab}) = -if\sum_{a=1}^{n}\psi_a + if\sum_{a<b}F_{ab}q_{ab} + \frac{p}{N}G_1(q_{ab}) + G_2(\psi_a, F_{ab}). \tag{7}$$

The functions G_1 and G_2 are given by the expressions

$$G_1(q_{ab}) = \log\prod_{a=1}^{n}\left(\int_K^{\infty}\frac{d\lambda_a}{2\pi}\int_{-\infty}^{\infty}dx_a\right)\exp\left(i\sum_a\lambda_a x_a - \frac{1}{2}\sum_a x_a^2 - \sum_{a<b}q_{ab}x_a x_b\right) \tag{8}$$

$$G_2(\psi_a, F_{ab}) = \log\left[\sum_{\{J^a\}}\exp\left(i\sum_a\psi_a(J^a)^2 - i\sum_{a<b}F_{ab}J^aJ^b\right)\right]. \tag{9}$$

When N, Z and p tend to infinity while keeping the ratios $f=(N-Z)/N$ and $\alpha=p/N$ fixed, we can use steepest-descent methods to evaluate the integral (7). This yields a set of equations for the stationary points of $G_n(\psi_a, F_{ab}, q_{ab})$. In order to solve these equations, we look for a replica-symmetric solution where

$$\psi_a = \psi \qquad F_{ab} = F \qquad q_{ab} = q \tag{10}$$

independent of the index a or b. Under this symmetry assumption it becomes easy to calculate the functions $G_1(q)$ and $G_2(\psi, F)$ exactly. On the other hand, in order to obtain the entropy from (6) we must consider the limit $n\to 0$. In this limit, the three stationary-point equations become

$$f = \int_{-\infty}^{\infty} Dz\,\frac{\exp[-(\phi+\frac{1}{2}q')]2\cosh(\sqrt{q'}\,z)}{1+\exp[-(\phi+\frac{1}{2}q')]2\cosh(\sqrt{q'}\,z)} \tag{11}$$

$$fq = \int_{-\infty}^{\infty} Dz\left(\frac{\exp[-(\phi+\frac{1}{2}q')]2\sinh(\sqrt{q'}\,z)}{1+\exp[-(\phi+\frac{1}{2}q')]2\cosh(\sqrt{q'}\,z)}\right)^2 \tag{12}$$

$$fq' = \frac{\alpha}{2\pi(1-q)}\int_{-\infty}^{\infty} Dz\,\frac{\exp\{-[(K+\sqrt{q}\,z)/\sqrt{1-q}]^2\}}{\{H[(K+\sqrt{q}\,z)/\sqrt{1-q}]\}^2} \tag{13}$$

where we have put $\psi = i\phi$ and $F = iq'$ in order to obtain real-valued parameters. The other notation is like in [1]

$$Dz = \frac{\exp(-z^2/2)}{\sqrt{2\pi}}\,dz \qquad H(x) = \int_x^{\infty} Dz. \tag{14}$$

For given values of α and f, the stationary-point equations (11), (12) and (13) determine the value of ϕ, q and q'. The quenched entropy is then obtained as

$$S_Q(\alpha, f) = f\phi + f\frac{qq'}{2} + \alpha \int_{-\infty}^{\infty} Dz \log H\left(\frac{K + \sqrt{q}\,z}{\sqrt{1-q}}\right) + \int_{-\infty}^{\infty} Dz \log\{1 + \exp[-(\phi + \tfrac{1}{2}q')]2\cosh\sqrt{q'}\,z\}. \tag{15}$$

3. Upper bound for $\alpha_c(f)$ from the annealed entropy

Before taking up the study of the solution of the three stationary point equations, we first derive an exact upper bound for the maximum storage capacity $\alpha_c(f)$. This is easily obtained from the annealed entropy

$$S_A(\alpha, f) = \lim_{N\to\infty} \frac{1}{N} \log\langle\Omega\rangle. \tag{16}$$

From the concavity of the logarithm

$$\langle\log\Omega\rangle \leqslant \log\langle\Omega\rangle \tag{17}$$

we get immediately

$$S_Q(\alpha, f) \leqslant S_A(\alpha, f). \tag{18}$$

As we know that S_Q must be positive when $\alpha < \alpha_c$ we obtain from (18) that also S_A must be positive for $\alpha < \alpha_c$.

The annealed entropy can easily be calculated exactly. In the limit $N \to \infty$ we get

$$S_A(\alpha, f) = \alpha \log H(K) + f\log 2 - f\log f - (1-f)\log(1-f). \tag{19}$$

For each fixed value of f, the entropy S_A is a decreasing function of α because the value of $H(K)$ is always smaller than 1. $S_A(\alpha, f)$ becomes zero when α is equal to

$$\alpha_A(f) = \frac{f\log 2 - f\log f - (1-f)\log(1-f)}{-\log H(K)}. \tag{20}$$

It follows immediately that $\alpha_A(f)$ is an exact upper bound for the storage capacity. The function $\alpha_A(f)$ is shown in figure 1 (broken curve) for the case $K = 0$. It attains its maximum value $\log 3/\log 2 = 1.58$ for $f = \frac{2}{3}$.

An upper bound for $\alpha_c(f)$ can also be obtained from information theory. Since each neuron can be connected to the other N neurons by $\binom{N}{Nf}2^{Nf}$ different coupling vectors $\boldsymbol{J}$ which contain $N(1-f)$ zeros, it is possible to store $\log(\binom{N}{Nf}2^{Nf})/\log 2$ bits of information per neuron. This number is therefore an upper bound for the maximum number p of patterns which can be stored. This upper bound coincides with $\alpha_A(f)$ in (20) when we put $K = 0$.

4. The critical storage capacity derived from the quenched entropy

We now turn to the solution of the stationary point equations (11), (12) and (13). For given values of α and f, we must find the solution for q, q' and ϕ. This can, in general, only be done numerically. Only in special limiting cases is it possible to obtain solutions by analytic methods.

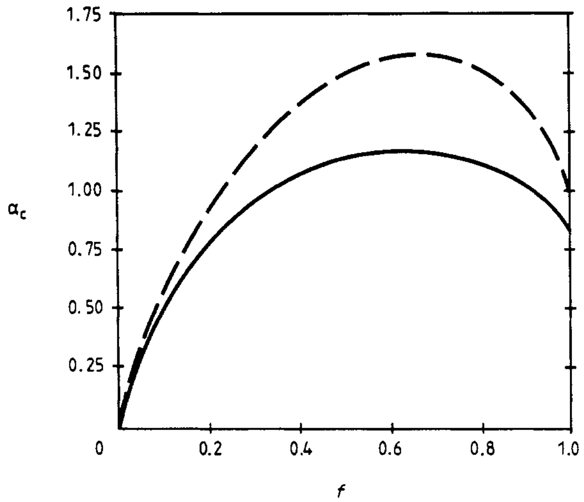

Figure 1. The storage capacity $\alpha_c(f)$ is given as a function of the fraction f of non-zero couplings (solid curve). The broken curve gives the upper bound $\alpha_A(f)$ as derived from the annealed entropy.

From equation (12) it follows that q must be positive and from equation (13) it follows that q must be smaller than 1. So q must lie in the interval (0, 1). At the lower limit $q=0$, we get $q'=0$ from (12) and $\alpha=0$ from (13). Since our goal is the determination of the maximum value of α, the neighbourhood of $q=0$ is uninteresting. Near the upper limit $q\rightarrow 1$, we can use asymptotic approximations which make the solution of the three equations much easier. This yields the largest value of α for which the three equations have a solution. But this solution again is uninteresting because it lies everywhere above the upper bound which we derived in the previous section. Like Gardner and Derrida [2], we have checked the condition of local stability for $q\rightarrow 1$ and have found that the replica-symmetric saddle point fails to satisfy this condition for all values of f.

Following Krauth and Mézard, we now determine the maximum storage capacity for each value of f from the condition that the quenched entropy must be non-negative. We use the replica-symmetric expression (15) for the entropy and determine $\alpha_c(f)$ by solving the equation

$$S_Q(\alpha_c, f)=0. \tag{21}$$

The solution of the four equations (11), (12), (13) and (21) for a given value of f can only be done numerically. For the case $K=0$, the resulting storage capacity $\alpha_c(f)$ is shown in figure 1 (solid curve). It lies everywhere below the upper bound $\alpha_A(f)$ as it should do if the inequality (18) is satisfied. The function $\alpha_c(f)$ has the same general form as $\alpha_A(f)$ and attains its maximum value 1.17 near $f=0.63$.

The calculated value of $\alpha_c(f)$ will be the exact storage capacity if the replica-symmetric expression (15) for $S_Q(\alpha, f)$ is exact. A necessary condition is that the replica-symmetric saddle point be locally stable at $\alpha_c(f)$. We have checked this condition and found it to be satisfied for all values of f. A more positive argument which suggests that the result for $\alpha_c(f)$ is probably correct comes from the generalisation of an elegant argument of Krauth and Mézard [3] in their calculation for the Ising case. This argument makes use of a result of Gardner's paper [1]. Considering the sphere of radius $\sqrt{N}$ in the space of interactions, Gardner determined the typical value of the fraction of the surface of the sphere where the p conditions (2) are satisfied.

The value of this fraction is equal to $e^{NG(q)}$ where the function $G(q)$ is given by Gardner's equation (20). The order parameter q depends on α via her saddle-point equation (Gardner's equation (23)). In the case of Ising couplings, there only exist 2^N coupling vectors and their endpoints lie distributed homogeneously on the surface of Gardner's sphere. Krauth and Mézard argue that it is reasonable to assume that α_c should be obtained when the fraction of the surface has become so small that on the average only one endpoint of the 2^N vectors lies in it. This determines α_c from the equation

$$\exp[NG(q(\alpha_c))] = \frac{1}{2^N} \tag{22}$$

or

$$G(q(\alpha_c)) = -\log 2. \tag{23}$$

Solving this equation, Krauth and Mézard find $\alpha_c = 0.85$, just a little above the value 0.83 which they obtained from the quenched entropy. It is easy to generalise their reasoning for diluted Ising couplings at least in the case $K = 0$. In this case the special value $\sqrt{N}$ of the radius of Gardner's sphere is irrelevant. For a general value of f, there are $\binom{N}{Nf}2^{Nf}$ coupling vectors with $N(1-f)$ zeros. The equation (23) is then replaced by

$$G(q(\alpha_c)) = -f\log 2 + f\log f + (1-f)\log(1-f). \tag{24}$$

Solving this equation together with Gardner's saddle-point equation yields a value for $\alpha_c(f)$ which runs parallel to and just above the storage capacity derived from the quenched entropy. The difference is so small that, if drawn in figure 1, it would hardly be distinguistable from the curve $\alpha_c(f)$.

5. Discussion

In this paper, we have generalised the calculation of the storage capacity by Krauth and Mézard to the case of diluted Ising couplings. Using the replica method with replica symmetry to calculate the quenched entropy, we have determined the storage capacity $\alpha_c(f)$ from the condition that the entropy must be non-negative. The result shows a 40% increase in storage capacity when one third of the couplings are cut in the fully connected Ising network. Although this may look attractive for practical applications, it is difficult to implement presently due to the lack of a good algorithm for determining the coupling coefficients.

There exist several reasons for believing that the result of the replica-symmetric calculation may be correct. It agrees closely with the value obtained from the attractive argument used by Krauth and Mézard and based on a formula of Gardner which is known to be correct. It also agrees qualitatively with exact numerical calculations by Vanderzande [9] for small systems of up to fifteen neurons for values of $f = \frac{1}{3}, \frac{1}{2}$ and 1. Finally for all values of f the replica-symmetric saddle point is locally stable at $\alpha_c(f)$. As a matter of fact we have checked that the stability condition of the saddle point is satisfied for even larger values of α, at least as large as $\alpha_A(f)$.

It is interesting to compare our results with those of Sompolinsky [7] and Van Hemmen [8] for a network with three-level synapses. Starting from the Hopfield model, all couplings are severed which are weaker than a chosen cut-off value x_0 while all remaining couplings are replaced by the Ising values ± 1 depending on their sign. The value of the parameter x_0 determines the overall degree of dilution. The fraction of non-zero bonds is equal to $c_0 = 1 - \mathrm{Erf}\ (x_0/\sqrt{2})$. From the procedure of cutting the couplings, it is clear that the position of the missing bonds is not random but correlated with the stored patterns. The calculated storage capacity α_c as a function of the connectivity parameter c_0 (the same as our f) displays the same general behaviour (Sompolinsky's figure 4.4) as our function $\alpha_c(f)$. The storage capacity goes through a maximum for a value of f smaller than 1. Sompolinsky and Van Hemmen obtain the maximum storage when $x_0 = 0.62$ which corresponds to $c_0 = 0.53$ (in Sompolinsky's paper, the value $c_0 = 0.63$ is quoted, but this seems to be a printing error). The maximum in our calculation occurs for $f = 0.63$ but as both curves are very flat near their maxima, they look quite similar.

From the calculated storage capacity $\alpha_c(f)$ we can easily determine the derivative $\alpha_c'(f)$. The physical meaning of this quantity is obvious: for any value of f it gives the possible increase in storage capacity due to a small increase in connectivity. More intuitively, this derivative is a measure for the efficiency of the added connections for storing additional patterns. From figure 1 it is seen that $\alpha_c'(f)$ is a monotonously decreasing function of f. For very small values of f, the derivative $\alpha_c'(f)$ has the same meaning as the ratio $\alpha_c(f)/f$ which is the storage capacity per synapse. This latter quantity has been considered in neural networks with strong random dilution [6] because it tends to a finite value when f tends to zero. In our calculation in which the position of the missing bonds is optimised, the value of $\alpha_c'(f)$ diverges logarithmically when f tends to zero:

$$\alpha_c'(f) \to -\frac{\log f}{\log 2} \qquad \text{for } f \to 0. \tag{25}$$

In a very diluted network, the couplings between the neurons are extremely efficient in storing many patterns. As an example, for $f = 10^{-8}$, the right-hand side of (25) is of order 26, indicating that one extra coupling per neuron could store 26 extra patterns. When the value of f increases, $\alpha_c'(f)$ decreases rapidly and becomes very small when each neuron is connected to half of the other neurons. A further increase in connectivity does not increase the storage capacity. Beyond $f = 0.63$ the efficiency even becomes negative, corresponding to a decrease of the global storage capacity.

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